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LETTERS TO THE EDITOR

FATIGUE ANALYSIS OF NON-LINEAR STRUCTURES WITH VON MISES STRESS

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1. INTRODUCTION

Fatigue analysis of vibrating structures subject to random excitations due to earthquakes, wind loadings, or ocean waves is an important subject [1–4]. When the excitation level is high, the non-linear behavior of the structural system has significant effect on the fatigue damage. However, when the system is non-linear, the fatigue analysis becomes very involved [5–7]. In an earlier study, we have developed a computationally practical method for predicting the fatigue life of non-linear structures under random excitations [7]. The method consists of an application of equivalent linearization combined with Monte Carlo simulations in a decoupled linear modal space, and can efficiently generate time histories of the stresses of the structure for the fatigue calculation. In that paper, the maximum stress of a plate in one particular direction was used in the fatigue calculation, although the plate was clearly in multi-axial stress state. Recently, the well-known von Mises stress has been used to estimate the fatigue damage of structures in the multi-axial stress state [8–12]. The work reported in these papers did not attack the nonlinear analysis of structures. In this paper, we shall extend the method developed in reference [7] to study the fatigue damages in non-linear structures by using the equivalent von Mises stress proposed in [9–12].

We shall use a non-linear rectangular von Kármán plate to demonstrate the analysis [13–17]. In the fatigue study, we can use the linear accumulative damage theory in conjunction with the S–N curve of the material [1, 18, 19]. Since the plate considered here has multiple resonant modes, the response will be broadband. It is well known that the Rain-Flow cycle counting scheme is well suited to a broadband process [2, 18, 20]. The present approach carries out the stress analysis in the time domain, and then applies the Rain-Flow scheme to count stress cycles. This fatigue estimate has to be done in the time domain because the Rain-Flow scheme is difficult to incorporate in a probabilistic formulation. The method of equivalent linearization is applied to solve for the response of the non-linear plate [6, 21–23]. The equivalent linearization method implies that the modal responses of the structure are Gaussian. We have shown that the estimated fatigue life of non-linear structures by the equivalent linearization method is in excellent agreement with the result obtained by direct numerical simulations [7, 24].

The remainder of the paper is organized as follows. In section 2, we present the method of equivalent linearization for calculating stresses of the non-linear plate. In section 3, we

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discuss the fatigue estimate by applying the classical fatigue theory along with the Rain-Flow cycle counting scheme to an equivalent von Mises stress of the plate.

2. EQUIVALENT LINEARIZATION OF NON-LINEAR PLATES

The present study uses the von Kármán plate theory [15]. Besides the typical assumptions of this theory, we point out that the in-plane inertia of the plate is neglected. After many steps of derivations, we arrive at a set of non-linear modal equations for the deflection of the plate [7]:

$$\ddot{W}_I + \xi\dot{W}_I + \frac{1}{M_I}\sum_{J=1}^{N} k_W(I,J)W_J + \frac{1}{M_I}\sum_{J=1}^{N}\sum_{K=1}^{N}\sum_{L=1}^{N} k_N(I,J,K,L)W_J W_K W_L = q_I\ddot{W}_0, \quad (1)$$

where $I = 1, 2, \ldots, N$, $k_N(I, J, K, L)$ is the non-linear stiffness matrix, M_I is the modal mass, and q_I is the modal force component, and $\ddot{W}_0$ is a Gaussian white noise base excitation to the plate. The modal expansions are given by

$$U(x,y,t) = \sum_{\tilde{I}=1}^{M} \alpha_i(x)\beta_j(y)U_{\tilde{I}}(t), \qquad V(x,y,t) = \sum_{\tilde{I}=1}^{M} \gamma_i(x)\eta_j(y)V_{\tilde{I}}(t),$$

$$W(x,y,t) = \sum_{I=1}^{N} \phi_i(x)\psi_j(y)W_{\tilde{I}}(t). \quad (2\text{–}4)$$

The spatial functions of x and y are chosen such that the geometrical boundary conditions of the plate are satisfied.

Since the probability distribution of the non-linear response of the plate is difficult to obtain, a sufficiently long time history of the response under random excitations is needed to estimate the fatigue life. Due to the triple summation of the non-linear term in equation (1), it is time-consuming to solve for the response in the time domain by using full numerical simulation methods when the number of terms N is large. The numerical effort in evaluating equation (1) for one time step is proportional to N^4.

Consider a linear system equivalent to equation (1),

$$\ddot{W}_I + \xi\dot{W}_I + \frac{1}{M_I}\sum_{J=1}^{N} k_e(I,J)W_J + e_I(\underline{W}) = q_I\ddot{W}_0, \quad (5)$$

where $k_e(I, J)$ is an equivalent linear stiffness matrix and the error $\mathbf{e} = \{e_I(\underline{W})\}$ is defined by

$$e_I(\underline{W}) = -\sum_{J=1}^{N} k_e(I,J)W_J + \sum_{J=1}^{N} k_W(I,J)W_J + \sum_{J=1}^{N}\sum_{K=1}^{N}\sum_{L=1}^{N} k_N(I,J,K,L)W_J W_K W_L. \quad (6)$$

The elements of $k_e(I, J)$ are chosen so as to minimize the steady state mean square value $\mathrm{E}[\mathbf{e}^{\mathrm{T}}\mathbf{e}]$ where $\mathrm{E}[\cdot]$ denotes the expected value. This will lead to an approximate solution to equation (1) in which the numerical effort for one time step is proportional to N^2. This represents substantial computational savings and makes it practical to include a large number of terms.

Since $\ddot{W}_0$ is Gaussian, the solution for W_I of the equivalent linear system is known to be jointly Gaussian and all the odd order steady state moments of W_I are zero. By using this property, we obtain an expression for the equivalent linear stiffness as

$$k_e(I,J) = k_W(I,J) + \sum_{K=1}^{N}\sum_{L=1}^{N}(k_N(I,J,K,L) + 2k_N(I,K,L,J))R_W(K,L), \tag{7}$$

where $R_W(K,L) = \mathrm{E}[W_K W_L]$ is the steady state correlation matrix of W_K. $R_W(K,L)$ is still unknown and one has to solve for $k_e(I,J)$ and $R_W(K,L)$ simultaneously. An iterative procedure can be developed to solve for these quantities [7].

Let $\mathbf{\Phi}(I,J)$ be the eigenmatrix whose columns are eigenvectors of the free, undamped system

$$\ddot{W}_I + \frac{1}{M_I}\sum_{J=1}^{N} k_e(I,J)W_J = 0. \tag{8}$$

$\mathbf{\Phi}(I,J)$ has the properties

$$[\mathbf{\Phi}]^{\mathrm{T}}[\mathbf{\Phi}] = [\mathrm{I}], \qquad [\mathbf{\Phi}]^{\mathrm{T}}[M]^{-1}[k_e][\mathbf{\Phi}] = [\Omega^2], \tag{9}$$

where $[M]$ is the diagonal mass matrix with elements M_I, $[\mathrm{I}]$ is the unit matrix, and $[\Omega^2] = \mathrm{diag}(\omega_I^2)$ is the $N \times N$ diagonal matrix. ω_I is the Ith natural frequency of the system (8). Consider the transformation to the modal co-ordinates B_J,

$$W_I = \sum_{J=1}^{N}\mathbf{\Phi}(I,J)B_J, \qquad f_I = \sum_{J=1}^{N}\mathbf{\Phi}(J,I)q_J. \tag{10}$$

We obtain a set of uncoupled ordinary differential equations,

$$\ddot{B}_I + \xi\dot{B}_I + \omega_I^2 B_I = f_I\ddot{W}_0. \tag{11}$$

Let $G_{\ddot{W}}$ denote the single-side power spectral density of $\ddot{W}_0(t)$. The steady state correlation matrix for the modal co-ordinates B_J can be obtained as [1, 3]

$$R_B(I,J) = E[B_I B_J] = \frac{\xi f_I f_J G_{\ddot{W}_0}}{(\omega_I^2 - \omega_J^2)^2 + 2\xi^2(\omega_I^2 + \omega_J^2)}. \tag{12}$$

The correlation matrices $R_W(I,J)$ and $R_B(I,J)$ are related by

$$[R_W] = [\mathbf{\Phi}][R_B][\mathbf{\Phi}]^{\mathrm{T}}. \tag{13}$$

We can now state the iterative procedure to determine $[k_e]$ and $[R_W]$.

1. The zeroth order approximation: assume that $k_e(I,J) = k_W(I,J)$.
2. Determine the eigenvalue and eigenmatrix $[\mathbf{\Phi}]$ of equation (8).
3. Compute $R_B(I,J)$ and $R_W(I,J)$. Then, update $k_e(I,J)$ by using equation (7).
4. Check convergence. Let $[k_e]_n$ denote the equivalent linear stiffness matrix at the nth iteration. Stop the iteration if $\|[k_e]_n - [k_e]_{n-1}\| \leqslant \delta_k$, where $\|\cdot\|$ denotes the norm of a matrix; otherwise, repeat the iteration from Step 2. In the present study, the infinite norm $\|[R]\|_\infty = \max_{ij}|R_{ij}|$ is used.

The stress components of the plate are given by

$$\sigma_{xx} = \frac{E}{1-v^2}\left[\sum_{\tilde{I}=1}^{M}(\alpha_i'(x)\beta_j(y)U_{\tilde{I}} + v\gamma_i(x)\eta_j'(y)V_{\tilde{I}}) - z\sum_{I=1}^{N}(\phi_i''(x)\psi_j(y) + v\phi_i(x)\psi_j''(y))W_I \right.$$
$$\left. + \frac{1}{2}\sum_{I=1}^{N}\sum_{J=1}^{N}(\phi_i'(x)\psi_j(y)\phi_k'(x)\psi_l(y) + v\phi_i(x)\psi_j'(y)\phi_k(x)\psi_l'(y))W_I W_J\right],$$

$$\sigma_{yy} = \frac{E}{1-v^2}\left[\sum_{\tilde{I}=1}^{M}(v\alpha_i'(x)\beta_j(y)U_{\tilde{I}} + \gamma_i(x)\eta_j'(y)V_{\tilde{I}}) - z\sum_{I=1}^{N}(\phi_i(x)\psi_j''(y) + v\phi_i''(x)\psi_j(y))W_I \right.$$
$$\left. + \frac{1}{2}\sum_{I=1}^{N}\sum_{J=1}^{N}(\phi_i(x)\psi_j'(y)\phi_k(x)\psi_l'(y) + v\phi_i'(x)\psi_j(y)\phi_k'(x)\psi_l(y))W_I W_J\right], \tag{14}$$

$$\sigma_{xy} = \frac{E}{2(1+v)}\left[\sum_{\tilde{I}=1}^{M}(\alpha_i(x)\beta_j'(y)U_{\tilde{I}} + \gamma_i'(x)\eta_j(y)V_{\tilde{I}}) - 2z\sum_{I=1}^{N}\phi_i'(x)\psi_j'(y)W_I \right.$$
$$\left. + \sum_{I=1}^{N}\sum_{J=1}^{N}\phi_i'(x)\psi_j(y)\phi_k(x)\psi_l'(y)W_I W_J + \sum_{I=1}^{N}\sum_{J=1}^{N}\phi_i'(x)\psi_j(y)\phi_k(x)\psi_l'(y)W_I W_J\right],$$

where $U_{\tilde{I}}$ and $V_{\tilde{I}}$ are the modal coefficients of the in-plane displacements, which are related to the modal coefficients of the deflection [7]. Note that the stress of the plate is biaxial.

3. FATIGUE ESTIMATES

The S–N curves as a material property are often obtained under uniaxial stress conditions. It is quite natural to use the maximum stress in a particular direction of the plate to calculate the fatigue damage, as has been done in a previous study [7]. When the structure is in the biaxial or multiaxial stress state, we should consider other stress variables for calculating the fatigue damage.

To this end, we can look into the yield criterion in plasticity [25]. There are two criteria defined by two well-known stresses: Tresca and von Mises. The Tresca criterion is based on the assumption that the material failure occurs in pure shear, and the Tresca stress is the maximum shear stress. While there is less evidence of the correlation of the fatigue damage to the Tresca stress, there is evidence that the von Mises stress correlates well to the fatigue damage [10]. This may explain the recent interest in using the von Mises stress in the fatigue analysis of structures in the multiaxial stress state [9–12]. In the following, we shall adopt the equivalent von Mises stress developed by Preumont in the papers cited here for the fatigue damage estimate. We would like to point out also that the present study can be readily extended to the Tresca stress should there be materials found to obey the Tresca criterion in the fatigue process.

For the plate, the stress is biaxial. Let us define a stress vector as $\sigma = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy})^{\mathrm{T}}$. The equivalent von Mises stress σ_c can be written as

$$\sigma_c^2 = \sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2 = \sigma^{\mathrm{T}}[Q]\sigma = \mathrm{Trace}\{[Q][\sigma\sigma^{\mathrm{T}}]\}, \tag{15}$$

where

$$[Q] = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

From equation (15), we can derive the power spectral density (PSD) function of the equivalent von Mises stress as

$$\mathbf{\Phi}_c(\omega) = \text{Trace}\{[Q][\mathbf{\Phi}_{\sigma\sigma}(\omega)]\} \tag{16}$$

where $\mathbf{\Phi}_{\sigma\sigma}(\omega)$ is the PSD matrix of the stress vector σ. In the steady state, the expansion coefficients W_I for the deflection of the plate are assumed to be independently Gaussian. With this knowledge and the stress expressions (14), we can either numerically or analytically obtain $\mathbf{\Phi}_{\sigma\sigma}(\omega)$. Once $\mathbf{\Phi}_c(\omega)$ is determined, we can apply the stochastic simulation algorithm [26] to generate a large number of time histories of the von Mises stress. Note here that the von Mises stress generated in this manner is cyclic, and can be combined with the Rain-Flow cycle counting scheme to calculate the fatigue damage based on the classical linear theory for high cycle fatigue [9–12].

The material fatigue property is characterized by the S–N curve defined as [1, 18, 19]

$$N = c/S^b, \tag{17}$$

where S is the stress amplitude, N is the number of cycles to failure. According to the Palmgren–Miner linear damage accumulation rule, the total damage D is obtained as the sum of accumulative damages caused by each cyclic damage event ΔD_i: $D = \sum_i \Delta D_i$. Failure is said to occur when $D = 1$. The fatigue life is then the amount of time it takes for this to happen. Based on the S–N curve, the damage increment due to one cycle is $\Delta D_i = |S_i|^b/c$, where S_i is the stress amplitude of the ith damage event determined by the cycle counting scheme.

Since the bandwidth of the response of the multi-mode non-linear plate is wide, we have to use the Rain-Flow cycle counting scheme for identifying damage events [2, 18, 20]. Once the fatigue damage is accumulated for a sufficiently long time τ, the average damage rate is given by

$$\Delta = D/\tau, \tag{18}$$

and the simulated average fatigue life is approximately determined as

$$T = 1/\Delta. \tag{19}$$

In general, ΔD_i and T are random variables with unknown probability distributions.

It should be noted that the iteration process for determining the equivalent linear system takes a very small fraction of the total computation time. Once the equivalent linear system is determined, the computational effort for simulating, the time series of the equivalent von Mises stress is independent of the type of non-linearity. The computational effort is thus significantly reduced as compared to the direct simulation of the full non-linear system.

4. CONCLUSIONS

This paper presents a method for calculating the fatigue damages of a non-linear plate subject to random excitations. The equivalent linearization is used for obtaining the response of the plate. An equivalent von Mises stress can be used in connection with the Rain-Flow cycle counting scheme to estimate fatigue damages based on the classical linear accumulative damage theory and the S–N curve of the material. A large number of time histories of the von Mises stress can be generated efficiently by making use of its PSD function.

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